A generic unified reinforced concrete model

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The behaviour of reinforced concrete members with ductile steel reinforcing bars at the ultimate limit state is extremely complex. Consequently, there has been a tendency for the seemingly disparate research areas of flexure, shear and confinement to follow separate paths in order to develop safe approaches to design. In this paper, it is shown how the already much researched and established, but somewhat peripheral, areas of reinforced concrete research of shear friction, partial interaction and rigid body displacements can be combined to produce a single unified reinforced concrete model that simulates the moment–rotation of hinges and their capacities, the shear deformation across critical diagonal cracks leading to failure and the effect of confinement on these behaviours. It is shown that this unified reinforced concrete model is completely generic, as it can be used to simulate reinforced concrete members with any type of reinforcement material (including brittle steel or fibre-reinforced polymer), various cross-sectional shapes of reinforcement (not only round bars but also flat externally bonded or rectangular near surface mounted adhesively bonded plates) and any type of concrete (e.g. high-strength or fibre-reinforced concrete). This new model, therefore, should allow the development of more accurate and safe design procedures as well as enabling more rapid development of new technologies.

Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>A&lt;sub&gt;sl&lt;/sub&gt;</td>
<td>cross-sectional area of longitudinal reinforcement</td>
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<tr>
<td>A&lt;sub&gt;reinf&lt;/sub&gt;</td>
<td>cross-sectional area of reinforcement</td>
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<tr>
<td>a</td>
<td>length of shear span</td>
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<tr>
<td>b&lt;sub&gt;v&lt;/sub&gt;</td>
<td>width of beam</td>
</tr>
<tr>
<td>c</td>
<td>cohesive component of Mohr–Coulomb failure plane</td>
</tr>
<tr>
<td>d</td>
<td>effective depth of beam</td>
</tr>
<tr>
<td>d&lt;sub&gt;sec&lt;/sub&gt;</td>
<td>depth of compression concrete strain hardening</td>
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<tr>
<td>d&lt;sub&gt;prism&lt;/sub&gt;</td>
<td>cylinder diameter</td>
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<tr>
<td>d&lt;sub&gt;soft&lt;/sub&gt;</td>
<td>depth of wedge</td>
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<tr>
<td>E</td>
<td>modulus</td>
</tr>
<tr>
<td>(EI)&lt;sub&gt;cr&lt;/sub&gt;</td>
<td>flexural rigidity of cracked member</td>
</tr>
<tr>
<td>f&lt;sub&gt;c&lt;/sub&gt;</td>
<td>unconfined compressive cylinder strength</td>
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<tr>
<td>f&lt;sub&gt;cc&lt;/sub&gt;</td>
<td>confined compressive cylinder strength</td>
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<tr>
<td>h</td>
<td>crack width</td>
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<tr>
<td>h&lt;sub&gt;agb&lt;/sub&gt;</td>
<td>crack opening due to aggregate interlock</td>
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<tr>
<td>h&lt;sub&gt;fl&lt;/sub&gt;</td>
<td>effective height of flexural crack</td>
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<tr>
<td>h&lt;sub&gt;sh&lt;/sub&gt;</td>
<td>crack width due to shear</td>
</tr>
<tr>
<td>h&lt;sub&gt;soft&lt;/sub&gt;</td>
<td>transverse movement of compression wedge</td>
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<tr>
<td>h&lt;sub&gt;a&lt;/sub&gt;</td>
<td>crack opening due to sliding along smooth wedge</td>
</tr>
<tr>
<td>K&lt;sub&gt;x&lt;/sub&gt;</td>
<td>&lt;sub&gt;SR&lt;/sub&gt;</td>
</tr>
<tr>
<td>k&lt;sub&gt;a&lt;/sub&gt;</td>
<td>neutral axis depth factor</td>
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<tr>
<td>L</td>
<td>length of prism, beam or lever arm</td>
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<tr>
<td>L&lt;sub&gt;per&lt;/sub&gt;</td>
<td>perimeter length of failure plane</td>
</tr>
<tr>
<td>L&lt;sub&gt;soft&lt;/sub&gt;</td>
<td>length of wedge; length of hinge</td>
</tr>
<tr>
<td>M</td>
<td>moment</td>
</tr>
<tr>
<td>M&lt;sub&gt;ap&lt;/sub&gt;</td>
<td>hinge moment at limit of rotation</td>
</tr>
<tr>
<td>m</td>
<td>frictional component of Mohr–Coulomb failure plane</td>
</tr>
<tr>
<td>P</td>
<td>axial force in reinforcement; longitudinal force</td>
</tr>
<tr>
<td>P&lt;sub&gt;fl&lt;/sub&gt;</td>
<td>reinforcement force due to flexure</td>
</tr>
<tr>
<td>P&lt;sub&gt;sh&lt;/sub&gt;</td>
<td>reinforcement force due to shear</td>
</tr>
<tr>
<td>P&lt;sub&gt;soft&lt;/sub&gt;</td>
<td>maximum longitudinal force the wedge can resist</td>
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<tr>
<td>s</td>
<td>slip across interface</td>
</tr>
<tr>
<td>s&lt;sub&gt;slide&lt;/sub&gt;</td>
<td>sliding capacity of interface</td>
</tr>
<tr>
<td>s&lt;sub&gt;soft&lt;/sub&gt;</td>
<td>slip across wedge interface</td>
</tr>
<tr>
<td>T&lt;sub&gt;fl&lt;/sub&gt;</td>
<td>shear force across sliding plane</td>
</tr>
<tr>
<td>V&lt;sub&gt;c&lt;/sub&gt;</td>
<td>concrete component of the shear capacity</td>
</tr>
<tr>
<td>(V&lt;sub&gt;c&lt;/sub&gt;)&lt;sub&gt;pl&lt;/sub&gt;</td>
<td>shear capacity of concrete along inclined plane</td>
</tr>
<tr>
<td>V&lt;sub&gt;g&lt;/sub&gt;</td>
<td>shear force at support</td>
</tr>
<tr>
<td>w&lt;sub&gt;b&lt;/sub&gt;</td>
<td>width of the wedge</td>
</tr>
<tr>
<td>α</td>
<td>slope of weakest wedge</td>
</tr>
<tr>
<td>β</td>
<td>angle of inclined crack</td>
</tr>
<tr>
<td>Δ</td>
<td>slip of reinforcement at crack face</td>
</tr>
<tr>
<td>Δ&lt;sub&gt;ζ&lt;/sub&gt;</td>
<td>step change in curvature</td>
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<tr>
<td>δ&lt;sub&gt;h&lt;/sub&gt;</td>
<td>bond interface slip</td>
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1. Introduction

The development of a unified reinforced concrete model stems from the authors’ research on developing design rules for the new technology of fibre-reinforced polymer (FRP) and steel plating reinforced concrete and masonry structures (Oehlers and Seracino, 2004; Oehlers et al., 2008a; Seracino et al., 2007; Xia and Oehlers, 2006). The inclusion or practical application of new technology (Denton, 2007; Oehlers, 2010) is illustrated in Figure 1, which depicts the role of research in taking innovative ideas into practice. It is suggested that the main thrust of civil engineering research is to develop a safe general design for all failure mechanisms, as shown in box 1 of Figure 1. This can then be classified as current technology for use in established applications, one example of which is reinforced concrete with normal-strength concrete and ductile steel reinforcement that can be applied to standard bridges and structures under monotonic and seismic loads. This safe general design is often codified, but the emphasis in a national code or standard is to ensure that it is a safe design and not necessarily the most advanced or efficient design.

Historically, ductile steel reinforced concrete (box 2a in Figure 1) was considered to be a new technology and research was required to bring it to the level of a safe general design for observed or envisaged failure mechanisms. Where possible, sound structural mechanics models were developed that simulated and quantified the failure mechanisms (box 5a); as they were based on structural mechanics principles, they required relatively little testing to validate and calibrate. An example is the material capacities of reinforced concrete members subjected to flexural and axial loads and only for members with ductile steel reinforcement and normal-strength concrete such that the crushing of concrete at an effective strain $\varepsilon_c$ always governs failure (i.e. members always

<table>
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<tr>
<th>$\delta_{\text{max}}$</th>
<th>maximum interface slip that resists shear</th>
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<tbody>
<tr>
<td>$\varepsilon$</td>
<td>strain; strain profile</td>
</tr>
<tr>
<td>$\varepsilon_{\text{c}}$</td>
<td>concrete crushing strain</td>
</tr>
<tr>
<td>$\varepsilon_{\text{pk}}$</td>
<td>concrete strain at maximum strength</td>
</tr>
<tr>
<td>$\varepsilon_{\text{soft}}$</td>
<td>longitudinal strain in compression wedge</td>
</tr>
<tr>
<td>$\theta$</td>
<td>rotation</td>
</tr>
<tr>
<td>$\theta_{\text{cap}}$</td>
<td>hinge rotation at limit of rotation</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>partial interaction parameter</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>stress; stress profile</td>
</tr>
<tr>
<td>$\sigma_{\text{fract}}$</td>
<td>confinement stress at fibre-reinforced polymer fracture</td>
</tr>
<tr>
<td>$\sigma_{\text{lat}}$</td>
<td>confinement stress</td>
</tr>
<tr>
<td>$\sigma_{\text{c}}$</td>
<td>compressive stress normal to crack interface</td>
</tr>
<tr>
<td>$\sigma_{\text{soft}}$</td>
<td>stress in softening wedge; residual strength</td>
</tr>
<tr>
<td>$\sigma_{\text{start}}$</td>
<td>stress at commencement of wedge formation; peak strength</td>
</tr>
<tr>
<td>$\tau_b$</td>
<td>bond shear stress</td>
</tr>
<tr>
<td>$\tau_{\text{max}}$</td>
<td>maximum interface shear capacity</td>
</tr>
<tr>
<td>$\tau_n$</td>
<td>shear stress across sliding plane</td>
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<tr>
<td>$\chi$</td>
<td>curvature</td>
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fail by concrete crushing). Further examples are strut and tie modelling, based on equilibrium and the buckling mechanism.

Where sound structural mechanics models were not available due to the incredible complexity of reinforced concrete behaviour, a large amount of testing through experiments and/or numerical modelling had to be done to develop empirical models (box 6a) that would temporarily fill the knowledge gap, ensure safe design and allow the rapid application of the new technology at the time. Examples (box 6a) include: the moment–rotation ($M/\theta$) of hinges through the use of moment–curvature ($M/\gamma$) and empirical hinge lengths; moment redistribution, which depends on the neutral axis depth factor, $k_d$; the concrete component of the shear capacity, $V_c$; the effect of concrete confinement, $\sigma_{con}$. As can be imagined, the development of these empirical models required a very large amount of testing as they could only be used within the bounds of their testing regimes, but their development was absolutely essential in order to be able to develop a safe design.

Having converted reinforced concrete with ductile steel reinforcement and normal-strength concrete to a safe general design (as represented by boxes 1, 2a, 5a and 6a in Figure 1), the question is: what has to be done to bring in a new technology such as the use of brittle FRP reinforcement or fibre-reinforced concrete in box 2b? The easiest way is to convert existing sound structural mechanics models (box 5a), which were based on concrete crushing, to allow for the other reinforcement failure mechanisms of fracture or debonding (Oehlers, 2007) (as in box 5b) as these structural mechanics models are generic. However, the major problem in developing a new technology to the point where it can be used safely for design is in the very expensive and time-consuming testing that is required when sound structural mechanics models are not available (box 6b). Hence, the necessity for developing new structural mechanics models to replace existing empirical approaches, as in box 6c, is the subject of this paper. This paper deals with what might loosely be termed hinge regions – regions where large deformations occur through flexure and shear, which lead to failure, and where, because of these large deformations, concrete confinement has the greatest effect.

The aim of this paper is to show how the peripheral research areas of shear friction (SF), partial interaction (PI) and rigid body displacements (RBDs) (box 7 in Figure 1) can be combined to form a generic unified model for the behaviour of reinforced concrete beams at the ultimate limit state. Some direct validation of aspects of the unified model is given in this paper, although most of it, where theoretical results are compared with tests on beams, eccentrically loaded prisms, pull tests and confined concrete cylinders, is published elsewhere (Daniell et al., 2008; Haskett et al., 2009a, 2009b; Mohamed Ali et al., 2008a, 2008b; Oehlers et al., 2008b; Seracino et al., 2007). It is not the aim of this paper to give the precise material properties required for the analyses, but rather to demonstrate how the unified model can be used to explain behaviours that are known to occur in experimental tests but which current empirical models have difficulty explaining, and consequently to bring a clearer insight into reinforced concrete behaviours and further confidence in the validity of the approach. It may also be worth noting that, unlike numerical models such as finite-element models, because the unified model is based on structural mechanics models, it allows the development of closed-formed mathematical models of failure mechanisms (Haskett et al., 2009a, 2009b; Mohamed Ali et al., 2008b, 2008c), which will aid in the development of design rules. The new unified model is, in this paper, in terms of a single primary crack, because the structural mechanics models have been developed for this problem (Haskett et al., 2009a, 2009b; Mohamed Ali et al., 2008a, 2008c). It is realised, however, that more than one primary flexural crack can dominate; structural mechanics models (Chen et al., 2007) are being developed for this, and numerical simulations based on the concepts described in this paper have been developed for multiple cracks (Liu et al., 2007; Mohamed Ali et al., 2008a, Oehlers et al., 2005). This is the subject of further research.

The current empirical approaches or models used to quantify moment–rotation ($M/\theta$), the concrete component of the shear capacity ($V_c$) and the effect of lateral confinement ($\sigma_{con}$) are described first, as these behaviours are used to illustrate the generic unified model. There already exists a great deal of very good research in what has been termed 'peripheral approaches' (box 7 in Figure 1) as they are not often used directly in design. These peripheral approaches are:

(a) shear friction (SF), sometimes referred to as aggregate interlock, which quantifies the behaviour across a crack interface or sliding plane

(b) partial interaction (PI), where slip occurs at an interface causing a discontinuity in the strain profile

(c) rigid body displacements (RBDs) such as in the opening up or sliding of crack faces.

The fundamental principles behind these peripheral approaches are described and then they are combined to form a new integrated failure mechanism that is the basis for the development of a unified model for the moment–rotation behaviour of reinforced concrete hinges. It is then shown how this unified $M/\theta$ model can be extended to allow for shear deformations and, interestingly, how shear failure depends on flexure. Finally, it is shown how the unified $M/\theta$ model can be used to quantify the effect of confinement in cylinder tests and, more importantly, in rectangular flexural members. But first, let us define the generic unified model and the fundamental principles on which it is based.

2. The generic unified reinforced concrete model

The generic unified reinforced concrete model is based on the assumption that the deformation of a reinforced concrete member can be attributed to both the variation in curvature between flexural/shear cracks (such as $\chi$ in the strain profile, $\varepsilon$, in Figure 2)
and the discrete rotation at each flexural/shear crack (shown as $2\theta$ in Figure 2). Between flexural/shear cracks, a Bernoulli linear strain profile (as in the strain profile, $\epsilon$) can be assumed; that is, there is no slip between the reinforcement and the concrete so that the strain profile is continuous as shown. This is often termed a full-interaction region. The deformation of the member due to curvature in this region can be derived using existing standard and well-established techniques that integrate the curvature along the member. In contrast, at flexural/shear cracks, a sudden discrete increase in rotation occurs as a consequence of crack widening due to the slip, $h$, between the reinforcement and the concrete crack faces. In this PI region, the opposing crack faces rotate relative to each other by $2\theta$. Being a discrete rotation at a specific section between crack faces, curvature simply does not apply at this section. This is the problem that needs solving. The three fundamental and well-researched and established principles of RBD, SF and PI are required to simulate and quantify the discrete rotation.

2.1 The three fundamental principles of the unified approach

RBD is defined as the relative movement of opposing concrete faces. Figure 3(a) shows the RBD of crack faces A–A and B–B when subjected to flexure. This flexural RBD can be quantified through the linear variation in crack width, $h$, that is the rotation,

$$2\theta.$$ Figure 3(b) shows the RBD due to shear, which causes sliding, $s$, and separation, $h$, as shown. For initially cracked sliding planes, the separation, $h$, is simply the crack width as shown. For initially uncracked sliding planes, $h$ is the separation of the sliding planes, which may be caused through the formation of a crack as in Figure 3(b) or the formation of herringbone cracks that rotate to cause the separation, $h$, and slide, $s$. The RBD can, therefore, be quantified through the rotation, $\theta$, separation, $h$, and slide, $s$.

Shear friction or aggregate interlock is defined here as the study of the relationship between the forces and displacements across potential sliding planes as in Figure 3(b). In other words, it is the relationship between the normal force, $P_n$, the shear force, $T_n$, the crack width, $h$, and the slide, $s$. The capacity of this interaction is often given in terms of a shear capacity stress, $v_u$, for a specific normal stress, $\sigma_n$:

$$P_n, T_n, h, s$$

with a capacity $v_u$.

Partial interaction is defined here as the study of the restraint across opposing faces due to reinforcement crossing the faces, as shown in Figure 4. The opposing faces in Figure 4 can only separate by $h$ as shown if there is slip between the reinforcement and each face, $\Delta$. For this slip at the face, $\Delta$, to occur, the reinforcement bond, $t_{rs}$, must depend on the interface slip between the concrete and the reinforcement, $\delta_h$; that is, there is not full interaction but PI. PI theory defines the relationship between the reinforcement force, $P_{reinf}$, and the slip at the face, $\Delta$.

It can be seen in the RBD in Figure 4 that there is a direct relationship between the separation, $h$, which is a component of SF, and the reinforcement slip, $\Delta$, which is a component of PI theory. Hence the necessary and direct connections between the three fundamental principles of RBD, SF and PI are used in this
paper to quantify and illustrate a generic unified reinforced concrete model.

2.2 Generic unified reinforced concrete model

mechanism

In the remainder of this paper, the separation, $h$, between sliding planes will be referred to as the crack width whether or not the potential sliding planes were initially cracked or not. The flexural response of a beam with an inclined flexural/shear crack is shown in Figure 5. The force in the tension reinforcement, $P_{fl}$, required for equilibrium causes the slip, $\Delta s$, through PI and consequently a crack width, $h_t$. Hence the tension region is governed by PI theory.

Softening of the compression region in Figure 5 is caused by the formation of compression wedges, which can eventually disconnect from the main body of the beam by uncontrollable sliding along the wedge interface. This is a classical SF problem as indicated on the right-hand side of the wedge in Figure 5. Sliding at the wedge interface $s$ causes crack widening, $h_s$, and, hence, the stirrups crossing this sliding plane on the left-hand wedge have to slip $\Delta s$, which induces $P_s$ to accommodate $h_s$. Hence in this region SF induces PI.

In Figure 6, the beam in Figure 5 that was originally subjected to flexure is now also subjected to shear. In the tension region, shear causes sliding, $s$, which further widens the crack by $h_{sh}$, causing an increase in the reinforcement force of, $P_{sh}$. This increase in the reinforcement force, $P_{sh}$, is balanced by a force across the crack of $P_{sh}$ as shown in Figure 6 such that the resultant force remains at $P_{fl}$ and its position remains the same. Hence, this is an SF PI region. The same mechanism occurs in the compression region. In this case, there is already a compressive force across the sliding plane of magnitude $P_{fl}$ from the flexural deformation in Figure 5 so that the force in the compression reinforcement, $P_{sh}$, in Figure 6 is induced by crack widening through SF.

Figure 5 shows that as flexure induces a tensile force in the reinforcement of $P_{fl}$ there must be an equal and opposite compressive force, $P_{fl}$, that is offset by the lever arm, $L$, to maintain rotational equilibrium. In contrast, in Figure 6 in the tension region, shear deformations induce equal tensile and compressive forces of $P_{sh}$ as shown, but with a lever arm of zero length. This, although obvious, is worth stressing as it is the essential difference between flexural PI behaviour and shear PI behaviour and is important in developing the unified approach.

3. Current empirical approaches

The empirical models noted in box 6a in Figure 1 will now be described to explain the need for structural mechanics models. Empirical models are frequently dimensionally incorrect but this is certainly not of any major concern as they are there to ensure safe design. Special attention will be paid in the following section to current moment–rotation approaches as it will be shown later that it is the moment–rotation behaviour that controls the shear behaviour as well as that due to confinement.
3.1 Moment–rotation
An example of a slab subjected to a blast load (Wu et al., 2007) and which failed in flexure is shown in Figure 7. The slab can be divided into non-hinge and hinge regions, as shown in Figure 8. The non-hinge region generally encompasses most of the member and is associated with narrow flexural cracks and where the concrete in compression is in its ascending or first branch, as in the stress profile, $\sigma_{\text{non-hinge}}$. Analysis of this non-hinge region is straightforward, as standard methods of equilibrium and compatibility apply. It is complex where flexural cracks occur as PI, through reinforcement slip, and disturbed regions have to be accommodated (Liu et al., 2007; Mohamed Ali et al., 2008a; Oehlers et al., 2005). However, as the number of cracks and crack widths are small, methods based on standard moment–curvature approximations (as in the $\sigma_{\text{non-hinge}}$ profile) generally suffice. In contrast, the hinge region occurs over a small length, is associated with concrete softening in a wedge and is also associated with wide cracks where the discrete rotation, $\theta$, is concentrated and in particular where the permanent rotation is concentrated. It is the moment–rotation of the hinge region that is considered here and not that of the non-hinge region where standard procedures can be applied.

Close-up views of typical hinges are shown in Figure 9: the hinge in (a) was formed in a reinforced concrete slab made with steel fibre concrete and subjected to a blast load; the hinge in (b) was formed in a steel-plated beam subjected to monotonic load (Oehlers and Seracino, 2004). The three-dimensional (3D) moment–rotation response of flexural members is required in design to quantify things such as the column drift, moment redistribution.
and the ability to absorb energy from dynamic loads. A common approach is to use a full-interaction sectional analysis to derive the moment–curvature relationship (Oehlers, 2007) (an outcome of which is shown as \( \Theta_{\text{hinge}} \) in Figure 8) and then to integrate this over a hinge length to get the moment–rotation relationship; this will be referred to as the moment–curvature hinge length approach.

For ease of discussion, let us idealise the moment–curvature relationship as bilinear (Oehlers, 2006), as in Figure 10. Let us assume that the cross-section of the continuous beam in Figure 11(a) has the bilinear relationship O–A–B in Figure 10, which has a rising branch O–A with a peak moment, \( M_A \), that occurs at a curvature, \( \chi_A \), followed by a falling branch A–B. On initially applying the load to the beam in Figure 11(a), all cross-sections of the beam over the whole length follow the rising branch O–A in Figure 10 until the moment at the supports just reaches \( M_A \) at a curvature, \( \chi_A \), as shown in Figure 11(b). Hence there are only two points in the beam at the supports where the moment \( M_A \) has been achieved; these are referred to as the hinge points in Figure 11(b).

Upon further downward deformation of the beam in Figure 11(a), the peak moment at the supports, \( M_A \), in Figure 11(b) must reduce as the curvature changes from \( \chi_A \) in Figure 10 to accommodate the deformation as any change in curvature from \( \chi_A \) in Figure 10 must result in a reduced moment as \( \chi_A \) occurs at the peak moment. Let us assume that the support moment reduces to \( M_1 \) in Figure 10, resulting in the distribution of moment in Figure 11(b) with a maximum \( M_A \). For this to occur, the curvature at the hinge points increases along the falling branch A–B in Figure 10 to \( \chi_{\text{BA}} \) to accommodate the deformation, as these are the only points that initially lay on the falling branch A–B. In contrast, the remainder of the beam (i.e. not including the hinge points) that was wholly on the rising branch O–A remains in the rising branch O–A, so that the curvature adjacent to the hinge points reduces to \( \chi_{\text{OA}} \). Hence there is a step change in the curvature, shown as \( \Delta\chi_0 \) in Figure 10, which occurs over a hinge of zero length as shown in Figure 11(c). This is, of course, an impossibility, as first recognised by Barnard and Johnson (1965) and Wood (1968), and this will be referred to here as the zero hinge length problem (Daniell et al., 2008; Oehlers, 2006).

To overcome the zero hinge length problem, which is peculiar to the moment–curvature approach, and in order to allow a safe design, the hinge length has been derived empirically (Baker, 1956; Corley, 1966; Mattock, 1967; Panagiotakos and Fardis, 2001; Priestley and Park, 1987; Sawyer, 1964); the exception is a more advanced technique, developed by Fantilli et al. (2002), which still uses a moment–curvature approach but which bases the hinge length on the softening wedge size, \( L_{\text{soft}} \), in Figure 8.

This approach of combining a two-dimensional (2D) moment–curvature analysis with an empirically derived hinge length is mathematically convenient and no doubt gives good results within the population of test results from which it was derived. However, it can give very large scatters (Panagiotakos and Fardis, 2001) when used outside the population of test results from which they were derived.

It is also worth noting that a moment–curvature analysis is a 2D analysis that is being used to simulate 3D moment–rotation behaviour. Take, for example, the hinges in Figure 9 where it can be seen that most of the rotation is concentrated in the flexural crack face rotation. The flexural cracks can only widen if there is slip between the reinforcement (for example, the longitudinal reinforcing bars in (a) represented by broken lines or the plate in (b)) and the concrete. If the bond between the reinforcement and concrete is strong and stiff then, for a given force in the reinforcement, the interface slip will be small and, consequently, the cracks will be narrow and the rotation small. Conversely, a weak bond will lead to large slips and consequently large rotations. It can be seen that a 2D moment–curvature analysis simply cannot allow for this 3D behaviour. It is also worth noting that 2D moment–curvature analyses cannot allow for the 3D behaviour of the compression wedges as shown in Figures 7 and 9 and which is a significant component of the overall behaviour.
Further evidence of problems associated with the application of the moment–curvature approach is illustrated in Figure 10. Take, for example, the case where the falling branch A–B becomes less steep as in A–C. The step change in curvature, $\Delta \chi_B$, that the hinge of zero length theoretically has to accommodate must increase from $\Delta \chi_B$ to $\Delta \chi_C$ until, for a perfectly ductile member (i.e. where the ‘falling branch’ is horizontal as in A–D), the step change required in the curvature tends to infinity. This is simply not correct. To complete the discussion, it is often perceived that the hinges in member sections with rising second branches (such as A–E in Figure 10) are easier to determine or can be quantified. This is a false assumption, as the hinge length is now purely a property of the bending moment distribution (MD); that is, the hinge length is the region of the beam where the moment lies in the region from A to E. As further proof of the inadequacies of this approach, it can be seen that when the slope of A–E tends to zero (i.e. it tends to the perfect ductility response A–D), the hinge is zero length; where A–E tends to the elastic condition A–F, the hinge length tends to $L/2$. All of which is nonsensical.

The above discussions illustrate that 2D moment–curvature analysis cannot be used to quantify, through structural mechanics, the hinge length. This has been referred to as the zero hinge length problem. Therefore, the moment–curvature hinge length approach is not what may, at first glance, appear to be a generic structural mechanics model, but an empirical model with its associated restrictions. Hence, the need for a structural mechanics model of the moment–rotation.

### 3.2 Concrete component of the shear capacity

An example of shear failure of a beam without stirrups is shown in Figure 12. The concrete component of the shear capacity, $V_c$, is one of the most intractable problems in reinforced concrete and an extensive amount of experimental testing has been required to develop empirical models to ensure safe design. An example (SAA, 1988) is shown in the following equation

$$V_c = \left(1.4 - \frac{d}{2000}\right) \left(\frac{2d}{a}\right) \left(\frac{A_{st}f_{c}}{bd}\right)^{1/3}$$

where $d$ and $b$ are the effective depth and width of a beam, $a$ is the shear span and $A_{st}$ the area of the longitudinal reinforcing bars. Lack of understanding of the structural mechanics behind this extremely complex problem of shear failure is reflected in the need for a size effect as in the first parameter, a dimensionally incorrect stress component of the fourth parameter and the use of lower bounds to represent the design strengths.

It needs to be emphasised that there is nothing wrong with this empirical approach, which is essential in both developing a safe design and identifying the parameters that control shear failure. However, because of the empirical nature of Equation 1, it would be very risky to apply it directly for new technologies such as the use of high-strength or high-performance concrete or brittle FRP reinforcement as this lies outside the bounds of the tests from which Equation 1 was derived. It will be shown later that shear failure is simply another limit to the moment–rotation capacity of a beam.

### 3.3 Concrete confinement

For a long time, confinement of concrete in compression has been known to enhance the ductility of reinforced concrete members. As far back as 1928, Richart et al. (1928) produced an empirical expression for the effect of hydrostatic confinement on the compressive cylinder strength:

$$f_{cc} = f_c + 4.1\sigma_{lat}$$

where $f_{cc}$ is the confined strength, $f_c$ is the unconfined strength and $\sigma_{lat}$ is the hydrostatic confinement. An extensive amount of testing was carried out over the intervening 80 years to refine this empirical model and to adapt it for FRP confinement as in Figure 13(a). However, columns are rarely, if ever, designed or fail in pure compression. Consequently, the greatest difficulty has been to take this empirical research on confined concrete cylinders (Figure 13(a)) and apply it to real structures where moment exists (Figure 13(b)) and rectangular sections (Figure 13(c)) where confinement is not uniform. It will be shown that the formation and sliding of compression wedges (Figures 7 and 9) is a major contribution to the dilation of concrete. This allows dilation to be

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Figure 12. Shear failure of beam without stirrups
included in the moment–rotation behaviour, which leads to a structural mechanics solution.

4. Peripheral approaches
The peripheral approaches of RBD, SF and PI (box 7 in Figure 1) that are needed to develop a structural mechanics model for the empirical models in box 6a are now described. This section will also show how they can be combined to form a generic integrated failure mechanism that can be used to model the seemingly disparate behaviours of moment–rotation, shear failure and the effect of confinement.

4.1 Rigid body displacements
The RBDs due to flexure of the hinges in Figure 9 are shown idealised in Figure 14 for a reinforced concrete beam with both externally bonded plates and reinforcing bars. The crack faces are shown as straight lines for convenience to emphasise that the variation of crack width over the crack height, $h_{cr}$, is linear; this linear variation in crack width occurs irrespective of the shape of the crack face due to the RBD as shown in Figure 3(a). Slip between the reinforcements and the concrete, $\Delta_{rebar}$ and $h_{plate}$ in Figure 14, allows the crack width to increase from zero to $h_{rebar}$ and $h_{plate}$; the consequence of this is rotation, $\theta$, of the flexural crack, where most of the hinge rotation is now concentrated. The compression wedge, which has been much studied through research on eccentrically loaded prisms (Debernardi and Taliano, 2001) as in Figure 15, also exhibits RBDs through the slip, $s_{soft}$, and separation, $h_{soft}$, shown in Figure 14.

The RBD due to shear across the critical diagonal crack in Figure

![Figure 14. Rigid body deformations due to flexure](image-url)
12 is shown idealised in Figure 16. The shear displacement $s$ induces a separation of the crack faces, $h$, due to aggregate interlock.

### 4.2 Shear friction

Extensive research has been carried out on SF (sometimes referred to as aggregate interlock), which is the behaviour across a crack, such as the critical diagonal crack in Figure 12 (idealised in Figure 16), that is subject to a shear RBD. Mattock and Hawkins (1972) quantified the shear capacity, $v_u$, in terms of the normal force across the crack, $n$:

$$v_u = c + m \sigma_n$$

where $c$ is the cohesive component and $m$ is the frictional component of the Mohr–Coulomb failure plane (Figure 17) and which depends on whether the failure plane was initially cracked or initially uncracked but deformed as through the formation of a herringbone formation of cracks.

Walraven et al. (1987) studied the effect of discrete particles bearing on each other (as in Figure 16) for an initially cracked sliding plane and quantified the relationship between the interface slip, $s$, the crack widening, $h$, induced by $s$ through aggregate bearing, the interface normal force, $\sigma_n$, and the shear, $\tau_n$, as shown in Figure 18. As an example, if an interface is displaced as shown in Figure 16 by both a specific slip $s = 0.7$ mm and crack width $h = 0.4$ mm then, from Figure 18, it can be seen that there is only one combination of $\tau_n = 0.76$ N/mm² and $\sigma_n = 0.38$ N/mm² that can occur. The works of Mattock and Hawkins (1972) and Walraven et al. (1987) fully describe the interaction between the shear behaviours, $s$, $h$, $\tau_n$ and $\sigma_n$, and the shear capacity, $v_u$, across an initially cracked sliding plane. Recently, Haskett et al. (2010a) quantified the SF properties across an initially uncracked sliding plane.

### 4.3 Partial interaction

The third peripheral research area is PI theory, which can be defined as the behaviour when there is slip across an interface, $s$, as this produces a step change in the strain profile at the reinforcement/concrete interface. PI theory was first developed by
Newmark et al. (1951) for the study of composite steel and concrete beam behaviour (Oehler and Bradford, 1995, 1999) and more recently in the study of intermediate crack debonding of adhesively bonded plates (Taljsten, 1996; Yuan et al., 2004).

Whenever a crack of any description intercepts reinforcement (e.g. the reinforcing bar in Figure 9(a), the externally bonded plate in Figure 19, the near surface mounted (NSM) plate in Figure 20 or the externally bonded plates used for shear strengthening in Figure 21) slip must occur across the interface between the concrete and the reinforcement to allow the crack to widen. This is therefore a PI problem and the solution to this problem requires knowledge of the interface bond slip characteristics, which can be determined from experimental tests. Typical bond slip values are shown in Figure 22 for a ribbed reinforcing bar, a NSM plate and an externally bonded plate, where \( \tau_b \) is the interface shear stress and \( \delta_b \) is the interface slip. The relationship is often idealised in the generic bilinear form shown (Figure 22) or the idealised linear descending variation where, in both, the important characteristics are the peak shear stress, \( \tau_{\text{max}} \), and peak slip, \( \delta_{\text{max}} \).

As an example of the application of PI theory, the following convenient closed-form solutions can be obtained at the ultimate limit state using the idealised linear descending bond slip variation in Figure 22. For reinforcement that has a uni-linear stress–strain relationship (e.g. FRP or steel prior to yield), the structural mechanics relationship between the force in the reinforcement, \( P_{\text{reinf}} \), in Figure 14, and the slip of the reinforcement at the crack face, \( \Delta_{\text{reinf}} \), is given by (Haskett et al., 2009a):

Figure 19. Intermediate crack debonding of externally bonded plate across flexural crack

Figure 20. Intermediate crack debonding of NSM side plate

Figure 21. Intermediate crack debonding of externally bonded plate across the critical diagonal crack

Figure 22. Bond–stress/slip characteristics (EB, externally bonded plate; NSM, near surface mounted)
4. \[ P_{\text{reinf}} = \frac{\tau_{\text{max}} L_{\text{per}}}{\lambda_1} \sin \{ \arccos \left( 1 - \frac{\Delta_{\text{reinf}}}{\delta_{\text{max}}} \right) \} \]

where

5. \[ \lambda_1 = \left( \frac{L_{\text{per}} \tau_{\text{max}}}{\delta_{\text{max}} E_1 A_{\text{reinf}}} \right)^{1/2} \]

where \( L_{\text{per}} \) is the width of the failure plane perimeter (for round bars this is the circumference and for externally bonded plates the width of the plate), \( A_{\text{reinf}} \) is the cross-sectional area of the reinforcement and \( E_1 \) is the elastic modulus of the reinforcement material. It is also worth noting that \( \lambda_1 \) is the minimum anchorage length (shown as \( L_{\text{anch}} \) in Figure 14) required to ensure that \( P_{\text{reinf}} \) is at its maximum.

For reinforcement with a bilinear stress–strain relationship, such as steel after yield in which the initial elastic modulus, \( E_1 \), changes to the strain hardening modulus, \( E_2 \), at a stress, \( f_1 \), then the behaviour after \( f_1 \) is given by

6. \[ P_{\text{reinf}} = \frac{\tau_{\text{max}} L_{\text{per}}}{\lambda_2} \sin \{ \arccos \left( 1 - \frac{\Delta_{\text{reinf}}}{\delta_{\text{max}}} \right) \} + A_{\text{reinf}} f_1 \]

where

7. \[ \lambda_2 = \left( \frac{L_{\text{per}} \tau_{\text{max}}}{\delta_{\text{max}} E_2 A_{\text{reinf}}} \right)^{1/2} \]

where \( \Delta_1 \) is the slip at stress \( f_1 \), which can be derived from Equations 4 and 5, and \( \lambda_2 \) is the additional anchorage length required to accommodate strain hardening. For reinforcement that is not fully anchored, similar expressions that depend on the boundary conditions (Mohamed Ali et al., 2008c) are available.

4.4 Integrated failure mechanism

It can now be seen that if there is RBD across two crack faces as in Figure 16, then the widening of the crack through SF or aggregate interlock by \( h \), which is twice the slip of the reinforcing bar as shown in Figure 14, will induce forces in the reinforcing bars that can now be quantified through PI theory. The research of Mattock and Hawkins (1972) and Walraven et al. (1987) depends on knowing the normal force, \( \sigma_n \), in Figure 16. PI theory completed their research by quantifying the normal stress induced by the reinforcement crossing this shear interface and consequently the shear capacity that depends on this normal stress. However, PI theory has also allowed the research of Mattock and Hawkins (1972) and Walraven et al. (1987) to be used not only under shear displacements but also under the much wider application of shear and flexural RBDs as shown in Figure 23 which will be used in the following unified models for moment–rotation, shear and concrete confinement.

Of interest, it can be seen that the shear resisted directly by the dowel action of the reinforcement crossing the crack is not addressed directly in the integrated failure mechanism in Figure 23. Dowel action is a recognised component of shear resistance. However, any additional shear resisted by dowel action will be offset by the reduction in the aggregate interlock shear capacity due to the reduced axial capacity of the reinforcement after allowing for the stresses due to the dowel action.

5. Moment–rotation model

The rigid body rotations of the hinges in Figures 7 and 9 have been idealised in Figure 14. The flexural hinge can be considered to have the three distinct components shown in Figure 24:

(a) the SF concrete compression softening wedge of depth, \( d_{\text{soft}} \), which can resist a force, \( P_{\text{soft}} \), and which, through PI, slips by a value, \( S_{\text{soft}} \);
(b) the depth of concrete in the ascending portion of its stress–strain relationship, \( d_{\text{asc}} \), which resists a force, \( P_{\text{asc}} \), that can be obtained from standard procedures and where the peak strain is \( \epsilon_{pk} \) as shown;
(c) the PI tensile zone of height \( h_c \), associated with rigid body rotation of the crack faces and where the reinforcement forces are \( P_{\text{reinf}} \).

Let us first consider the tension zone in Figure 24. For a given height of crack, \( h_s \), and for a given rotation, \( \theta \), the slip in the reinforcement layers, \( \Delta_{\text{reinf}} \), is fixed so that the force, \( P_{\text{reinf}} \), in each reinforcement layer can be determined from Equations 4–7. The limits to this rotation occur when either the reinforcements reach their slip at fracture, which can also be obtained from Equations 4–6, or when they debond, which, as a lower bound, occurs at \( \delta_{\text{max}} \) in Figure 22. What is still required is the limit to this rotation due to concrete compression wedge failure and also the extent of the compression zone (i.e. \( d_{\text{soft}} + d_{\text{asc}} \)) as this fixes the crack height, \( h_c \), that is needed for determination of the rotation, \( \theta \), for a specific reinforcement slip, \( \Delta_{\text{reinf}} \).
The behaviour of the concrete wedges in Figures 7, 9 and 15 and required in the analysis in Figure 24 can be derived from Mattock and Hawkins (1972) SF theory represented in Figure 17 and Equation 3. For a given depth of wedge, $d_{\text{soft}}$, in Figure 25 and from SF theory (Oehlers et al., 2008b), the force the wedge can resist is given by the equilibrium equation:

$$P_{\text{soft}} = w_b d_{\text{soft}} \left( \frac{c + \sigma_{\text{lat}} \cos \alpha (\sin \alpha + m \cos \alpha)}{\sin \alpha (\cos \alpha - m \sin \alpha)} \right)$$

where $w_b$ is the width of the wedge, which is generally the width of the beam, $\sigma_{\text{lat}}$ is the lateral confinement that can be induced by
the FRP wrap (as in Figure 25) or by the stirrups, $m$ and $c$ are the SF material properties shown in Figure 17 and Equation 3, and $\alpha$ is the angle of the weakest plane in Figure 25, given by:

$$\alpha = \arctan[-m + (m^2 + 1)^{1/2}]$$

As $\alpha$ is known in Figure 25, the length of the softening wedge, $L_{\text{soft}}$, can be determined for a given value of $d_{\text{soft}}$. The difference in the concrete compressive strain at the wedge interface, which is the peak ascending branch strain, $\varepsilon_{pk}$, minus the strain in the softening wedge, $\varepsilon_{\text{soft}}$, is the slip strain across the wedge interface which, when integrated over the length of the hinge, $L_{\text{soft}}$, gives the following interface slip having a limit, $s_{\text{slide}}$, which is a material property that can be determined from tests (Mattock and Hawkins, 1972; Mohamed Ali et al., 2008b)

$$s_{\text{soft}} = (\varepsilon_{pk} - \varepsilon_{\text{soft}})L_{\text{soft}} \leq s_{\text{slide}}$$

It is this latter limit, $s_{\text{slide}}$, that limits the depth of the softening wedge and by so doing restricts the rotation. This is the reason why deep beams often reach their rotational limit when the wedge slides uncontrollably and why shallow beams or slabs tend to have enormous rotational capacities only limited by fracture of the reinforcement. Thus, the three limits to the rotation, shown on the left-hand side of Figure 26(a) are

(a) sliding of the wedge when $s_{\text{soft}} = s_{\text{slide}}$
(b) fracture of the reinforcement (i.e. when the reinforcement slip $\Delta$ reaches the slip at reinforcement fracture given by Equations 4–7)
(c) debonding of the reinforcement when the slip is at least $\delta_{\text{max}}$ in Figure 22.

To determine which of these limits comes first, let us consider the bilinear strain profile in Figure 26(b) where there is a linear variation in strain in the compression ascending region of depth, $d_{\text{asc}}$, and zero tensile strain in the cracked region of height, $h_{\text{cr}}$; this is a generally accepted strain profile at the ultimate limit state when the flexural cracks are closely spaced. For a given width of concrete, $L$, it can be seen that the bilinear strain profile can be converted to a bilinear RBD as shown and, importantly, the rotation within the compression ascending zone, $\theta$, is simply the crack rotation, $\theta$. It is this relationship that allows the interaction between the wedge deformation and crack rotation (Haskett et al., 2009b) as shown in Figure 26(c).

The moment–rotation analysis (Haskett et al., 2009b) is depicted in Figure 26(c). Importantly, it is based on a linear RBD A–B–C as opposed to the conventional linear strain profile shown as $\varepsilon$ in Figure 2. Starting with a small rotation, $\theta$, the RBD A–B–C in Figure 26(c) is moved up or down while maintaining $\theta$ until the axial forces, $P$ (from Equations 4–9) sum to zero, after which the moment can be taken for that specific rotation. The rotation is gradually increased to obtain the moment–rotation curve, bearing in mind that a limit occurs when either the wedge reaches its sliding capacity, $s_{\text{slide}}$, or a layer of reinforcement either debonds or fractures. A typical analysis (Haskett et al., 2009b) is compared with test results in Figure 27; in this case, the rotation was
limited by fracture of the reinforcing bar, which is therefore the ductility limit at $M_{\text{cap}}/\theta_{\text{cap}}$.

6. Moment redistribution

As an example of the application of the moment–rotation model, it is applied here to moment redistribution to explain the difficulty with current empirical approaches. A two-span continuous beam is shown in Figure 28: virtually all of the permanent rotation is concentrated in single cracks within the hinges. It would thus be expected that it is these crack rotations that will control most of the moment redistribution.

Let us define the moment redistribution factor, $K_{MR}$, as the moment redistributed as a proportion of the moment if there were no redistribution; hence $100K_{MR}$ is the commonly used percentage moment redistribution. Let us consider the case of a continuous or encastré beam of span, $L$, and flexural rigidity, $(EI)_{cr}$ (as shown in Figure 29) that is subjected to a uniformly distributed load. Hence, before the hinges are formed (i.e. in the elastic range), from structural mechanics, the hogging or negative maximum moments are twice the maximum sagging or positive moment. Let us assume that the beam is loaded so that the hogging moments first reach their yield capacity, which is $M_{A}$ in Figure 11(b) so that the sagging moment is $M_{A}/2$. At this point, the slope at the supports is zero so that any hinge at the supports does not have to accommodate any rotation. Any further loading that increases the sagging moment above $M_{A}/2$, and consequently redistributes moment, will require the slope at the hogging regions to rotate (i.e. the hinge rotates $\theta$). It can be seen that the capacity to redistribute moment depends directly on the capacity of the hinge to rotate (i.e. its rotation capacity, $\theta_{\text{cap}}$). The behaviour described above can be quantified through structural mechanics to give the moment redistribution capacity of the beam (Haskett et al., 2010a, 2010b; Oehlers et al., 2008c) for redistribution from the hogging (negative) to sagging (positive) region, $K_{MR}$, as

$$K_{MR} = \frac{\theta_{\text{cap}}}{M_{\text{cap}}/L + \theta_{\text{cap}}/2(EI)_{cr}}$$

where the rotation at the supports is $\theta_{\text{cap}}$ when the moment at the supports is $M_{\text{cap}}$; $M_{\text{cap}}$ and $\theta_{\text{cap}}$ are the limits to the moment–rotation, which can be derived from the moment–rotation analyses described above and an example of which is shown in Figure 27.

National standards usually quantify the moment redistribution, $K_{MR}$, using the neutral axis depth factor, $k_{u}$. Examples of these empirical approaches from four national standards are shown in Figure 30. In general, they have a bilinear variation characterised by large differences between empirical models. Results from the unified moment–rotation approach are also shown in Figure 30 for beams of varying depth (Oehlers et al., 2008c). These also exhibit a ‘bilinear’ variation where the first part of the curve at lower values of $k_{u}$ is governed by bar fracture and the second part by wedge sliding and between these curves there is discontinuity. It can be seen that the variations from the national standards are just one part of a family of curves and hence just using the neutral axis factor, $k_{u}$, will never provide an accurate analysis; this explains the large variations between the empirical models from national standards.

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Figure 27. Comparison with test results

Figure 28. Hinges in continuous beam

Figure 29. Moment redistribution from negative region

---

80 60 40 20 0
0 0.01 0.02 0.03 0.04 0.05 0.06
Rotation: rad
Moment: kN m
Fracture ($M_{\text{cap}}, \theta_{\text{cap}}$)
Theoretical
Experimental
Rotation capacity of hinges $M_{\text{cap}}$, $\theta_{\text{cap}}$

A generic unified reinforced concrete model
Oehlers, Mohamed Ali, Griffith, Haskett and Lucas
7. Shear deformation

The moment–rotation model described above and illustrated in Figure 26 for a vertical crack can just as easily be applied to inclined cracks as in Figure 31(a) where a moment, $P_{fl}d$, induces a force in the reinforcement of $P_{fl}$, which in turn requires bar slips to produce a crack width of $h_{fl}$ at the soffit. The same inclined crack is now only subjected to shear deformations, as in Figure 31(b), which, through aggregate interlock, opens the crack by $h_{sh}$. This induces axial tensile forces in the bars of $P_{sh}$ but, more importantly, an equal but opposite axial compressive force whose resultant is at the same position (although shown slightly offset in Figure 31(b) for clarity) and which provides the normal interface stress, $\sigma_n$, that is essential for resisting the shear as shown in Figures 16–18.

The flexural and shear deformations in Figures 31(a) and (b) have been combined in Figure 32. It can be seen that the shear deformation increases the crack width by $h_{sh}$ and subsequently increases the force in the reinforcement from $P_{fl}$ to $P_{fl} + P_{sh}$, but the resultant tensile force remains at $P_{fl}$ because of the compressive force, $P_{sh}$, across the interface so that the moment remains at $P_{fl}d$. From both Walraven and others’ research for initially cracked sliding planes illustrated in Figure 18 (Walraven et al., 1987) and Haskett and others’ research for initially uncracked sliding planes (Haskett et al., 2010c), the concrete component of the shear capacity across the inclined plane, $\left(V_c\right)_\beta$, increases with $P_{sh}$, but reduces with the crack width, $h$; that is, there are two opposing effects on the shear capacity. The shear capacity is thus

\[
\left(V_c\right)_\beta = K \left(\frac{P_{sh}}{h_{sh}}\right) \left(\frac{1}{h}\right)
\]
limited not only by the remaining strength of the reinforcement after allowing for flexure, which emphasises the importance of strain hardening, but also by the crack width, which depends on the bond and bar properties as illustrated in Figure 22 and by Equations 4–7. It can now be seen how flexure and shear interact; no matter what the inclination of the crack and that neither the shear capacity nor the flexural capacities are independent. It can also be seen that shear failure can be envisaged as simply the fourth limit to the moment–rotation behaviour after wedge sliding, and reinforcement debonding or fracture.

It is also worth noting that vertical stirrups or externally bonded plates (as in Figure 21 and shown as the transverse reinforcement in Figure 33) can be treated the same way as the longitudinal reinforcement in this model because the forces in the vertical or transverse reinforcement, \(P_{sh}\), also depend on the crack width due to flexure and shear and there is also an interface compressive force, \(P_{sh, h}\). However, there is a subtle difference in their effects.

The ability to resist shear across an interface has been shown (Haskett et al., 2010c; Mattock and Hawkins, 1972; Walraven et al., 1987) to depend on the compressive force normal to the interface, as shown in Figures 17 and 18. Hence the interface compressive forces, \(P_{sh}\), in Figure 33 have been resolved about the slope of the interface, which is at an angle, \(\beta\), as shown in Figure 34. An inclined FRP NSM plate as in Figure 35 is also shown in Figure 34 where it is assumed to be perpendicular to the crack face. The compressive interface forces, \(P_{sh}\cos \beta + (P_{sh})_{NSM} + (P_{sh})_{long}\sin \beta\), provide the interface normal stress, \(\alpha_{n}\). Hence, depending on the variation of \(\alpha_{n}\) and the crack width, \(h\), and from the research of Walraven et al. (1987) (Figure 18) and Haskett et al. (2010c), the concrete component of the shear capacity, \(V_{c}h\), can be determined. It can be seen that reinforcement placed perpendicular to the critical diagonal crack, such as the NSM plate, is the most efficient at increasing the concrete component of the shear capacity, \(V_{c}h\). However, \(V_{c}h\) must also provide the inclined forces \(P_{sh,h}\sin \beta\) and \(P_{sh,h}\cos \beta\) to maintain equilibrium with the tensile forces in the reinforcement. Hence, the concrete component of the shear capacity available to resist shear is reduced by \([P_{sh,h}\cos \beta - P_{sh,h}\sin \beta]\). Hence, from this integrated failure mechanism, it can be seen that the longitudinal reinforcement induces a component of force that reduces the concrete shear capacity and the vertical reinforcement induces a component that is beneficial; this explains the subtle difference in their effects (Lucas et al., 2010).

### 7.1 Effect of span–depth ratio on shear

The concrete component of the shear capacity of a beam is known to increase with reduced shear spans as can be seen in the second parameter (2\(d/a\)) in Equation 1. This increase in strength has often been associated with arching action, but this increase in shear capacity can also be explained by the unified model. The concrete component of the shear capacity in empirical models is usually determined from tests on beams with short shear spans, as in Figure 36, to try to ensure that shear failure precedes...
flexural failure. In these tests and because of the short shear spans, a critical diagonal crack usually extends from the applied load to the support, as shown. This beam with this critical diagonal crack was analysed using a numerical model based on the unified model and the shear spans were varied as shown to determine the effect of the span–depth ratio.

For a chosen angle, $\beta$, in Figure 36, the analysis consisted of first imposing a flexural crack deformation of maximum width, $h_{fl}$, in Figure 36, which is shown as O–A at $h_{fl} = 0.42$ mm in Figure 37 for an analysis in which the inclined crack, $\beta$, is at $45^\circ$. This crack width, $h_{fl}$, fixes $P_{fl}$ (Equations 4–7) and consequently the applied moment. The shear crack width, $h_{sh}$, in Figure 36 was then gradually increased, that is $h_{total}$ in Figure 37, which imposed the interface force, $P_{sh}$, in Figure 36; for each increment of $h_{sh}$, the shear capacity, $V_c$, in Figure 37 was determined to get the variation B–C–D. From Walraven et al. (1987), the shear capacity is proportional to $P_{sh}$ and inversely proportional to $h_{total}$. Hence, initially the shear capacity increases along B–C as the effect of increasing the interface force, $P_{sh}$, dominates. After a while, along C–D, the effect of the crack width, $h_{total}$, dominates, even though the interface force is increasing, to produce a decline in strength.

As the flexural crack, $h_{fl}$, in Figure 36 is reduced to, say, $h_{fl} = 0.24$ mm in Figure 37 (i.e. O–E), the force due to flexure in the reinforcement reduces and hence a greater proportion of the reinforcement’s strength is available to resist shear; as would be expected, the shear capacity increases as can be seen by comparing F–G–H with B–C–D. A further reduction in the flexural crack width to $h_{fl} = 0.06$ mm gives J–K–L. The shear capacities at K, G and C in Figure 37 are the capacities that are available at specific applied moments and, in this case, reducing moments from K to C. The results are plotted in Figure 38 as the line M–N. The results for crack angles, $\beta$, of $40^\circ$ and $35^\circ$ are also shown as lines P–Q and R–S, respectively.

Figure 38 shows the variation of the shear capacity available with increasing moment. The ratio between the applied shear and moment depends on the beam geometry and these are shown as
the loading lines O–T at \( ad = 1 \) (45°), O–U at \( ad = 1.3 \) (40°) and O–V at \( ad = 1.6 \) (35°). Where these loading lines intercept the capacities gives the shear to cause failure (i.e. \( V_c \) at points W, X and Y). It can be seen that steep crack inclinations such as at \( \beta = 45^\circ \) with strengths \( M–N \) are stronger than shallow inclinations such as at \( \beta = 35^\circ \) with strengths \( R–S \); this is one explanation of the so-called arching action. However, just as important, the loading line is steeper for steep inclinations of crack, which by themselves would produce higher shear capacities. Hence, there are two effects that produce the arching action. It can be seen that the unified model can by itself explain the so-called arching effect, thus providing further insight into shear behaviour.

8. **Concrete confinement**

It has been shown that the SF analysis in Figure 25 can be applied to the 2D analysis of wedges in beams or prisms under flexure, such as in Figures 7, 9 and 15. However, wedges also form in the 3D problem of confined cylinders and rectangular prisms under pure compression, as shown in Figures 13(a) and (c). The SF theory developed by Mattock and Hawkins (1972) (Figure 17) will be applied to the analysis of confined concrete cylinders (Figure 13(a)) as further evidence of the usefulness of the SF component of the integrated failure mechanism. However, the analysis of confined cylinders under pure compression is fairly academic as virtually all structures are designed to take moment as in Figure 13(b) and are often not circular. To provide a solution, it will be shown that the unified moment–rotation model can also be used to simulate dilation and consequently the effect of confinement due to stirrups as well as for FRP wrap and for the analysis of rectangular sections.

8.1 **Pure compression members**

The stress–strain relationship of hydrostatically confined concrete has the typical shapes (Candappa et al., 2001; Xie et al., 1995) O–A–E–D–C shown in Figure 39 and increases in magnitude with the lateral confinement, \( \sigma_{lat} \). The branch O–A can be considered to be a material property with the peak value at the start of softening, \( \sigma_{d1} \), at A, which depends on the amount of hydrostatic confinement. It is hypothesised that, after the peak stress, \( \sigma_{sat} \), is reached, wedges start to form and that the wedges are fully formed when the strength stabilises at the residual strength, \( \sigma_{soft} \). A typical variation of the stress–strain relationship for FRP confined concrete is shown as the broken line E1–E2–E3. For example, at point E2, the FRP fractures so the confinement stress is known and equal to that for the hydrostatic variation O–A2–E2–D2 (i.e. \( \sigma_{sat}^2 \)); that is, the FRP stress–strain relation is just the one point E2 of this variation. Hence, the axial stress at FRP fracture, \( \sigma_{sat}^2 \), will be expected to be equal to or greater than the residual strength, \( \sigma_{soft} \).

The circumferential wedge in Figure 40 encompasses the whole of the truncated cone. When the depth of the wedge, \( d_{win} \), is equal to the radius of the cylinder, \( d_{win}/2 \), (i.e. the failure plane shown as D–E), then the behaviour of the horizontal plane is fully governed by SF so that the strength is \( \sigma_{sat} \). SF theory has been applied to cylinders (Mohamed Ali et al., 2008b). When the wedge in Figure 40, which encompasses the whole of the interior truncated cone, occupies the whole of the horizontal truncated plane, then the stress at which this occurs is the residual strength, \( \sigma_{soft} \), of the cylinder. Through equilibrium and the SF capacities of Equations 3 and 9, it is given by

\[
\sigma_{soft} = \frac{c + \sigma_{lat} \cos \alpha (\sin \alpha + m \cos \alpha)}{\sin \alpha (\cos \alpha - m \sin \alpha)}
\]

which is virtually the same equation as for the wedge in a rectangular section given by Equation 8 and where the wedge angle, \( \alpha \), is given by Equation 9.

The residual strength in Equation 12 varies according to the amount of confinement, \( \sigma_{lat} \). Applying typical values for the Mohr–Coulomb properties in Figure 17 of \( m = 0.8 \) and \( c = 0.17f_c \), Equation 12 converts to

\[
\sigma_{soft} = 0.71f_c + 4.3\sigma_{lat}
\]
which is a remarkably similar form to the empirical equation of Richart et al. (1928) for the strength of confined concrete (Equation 2). However, it should be remembered that Richart and others’ equation measures the peak strengths, $f_{cc}$, shown as $C_{243}$ start in Figure 39, whereas the SF expression in Equation 11 is a measure of the residual strengths, $C_{243}$ soft, in Figure 39.

The theoretical SF residual strengths from Equation 12 have been compared (Mohamed Ali et al., 2008b) with tests (Candappa et al., 2001; Mander et al., 1988; Martinez et al., 1984; Xie et al., 1995) in which the residual strength and confinement were measured directly. The results are shown in Figure 41 for both hydrostatically restrained concrete and steel spirally confined concrete and show good correlation. They have also been compared with FRP confined concrete (Mohamed Ali et al., 2008b) and, as expected, they have a strength that is slightly greater than $C_{243}$ soft.

The results in Figure 41 are good, if not very good, bearing in mind the large scatter associated with confinement research and also bearing in mind that a single value of both $m$ and $c$ was used and research has shown some variation, as shown in Figure 17. It would therefore appear that SF provides a structural mechanics model for confined concrete in cylinders and gives further confidence in the procedure outlined in Figure 25. It is now a question of: how do we apply the research on confined concrete under uniform compression to the real problem of beams or columns that have moment? This will be dealt with in the next section.

8.2 Compression members subjected to flexure

The analysis in Figure 26 allows the moment–rotation behaviour and limits to be determined. Even though the wedge moves upwards as a consequence of the interface sliding, $s_{soft}$ in Figure 26(a), it was not necessary to include this movement as the analysis was primarily concerned with unconfined concrete. However, sliding across the wedge interface causes an upward or dilation movement of the wedge, as shown in Figure 42. The upwards movement consists of two components. If the wedge interface was smooth, then sliding across the interface at an angle, $\alpha$, would cause an upward movement, $h_\alpha$, as shown, which depends purely on the geometry of the wedge. However, aggregate interlock also causes an upwards movement, $h_a$, which can be derived from the research on initially uncracked concrete (Haskett et al., 2010c), which is similar in form to the research (Walraven et al., 1987, Figure 18) for initially cracked concrete. Hence, the wedge by itself causes a dilation with two components, $h_\alpha$ and $h_a$, in Figure 42 as well as the usual dilation due to the elastic material Poisson effect.

The moment–rotation analysis in Figure 26 is shown again in Figure 43 to illustrate the effect of confinement from both stirrups and FRP wrap. To do the analysis (Farrall et al., 2008) an additional iterative procedure to that described for Figure 26 has to be included. For a fixed depth of wedge, $d_{soft}$, in Figure 43(b),

Figure 40. Circumferential wedges

Figure 41. Hydrostatically and spirally confined concrete

Figure 42. Dilation components in a beam
guess the confinement, $\sigma_{lat}$. For the depth of wedge, $d_{soft}$, the force in the wedge, $P_{soft}$, from Equation 8 induces a slip, $s_{soft}$, from Equation 10. This slip, $s_{soft}$, produces an upward movement, $h_{prism}$, equal to $h_a + h_\alpha$. This is in effect a crack width in Figure 43. If the plate is unbonded, then the strain in the plate due to the movement, $h_{prism}$, gives the stress and consequently the force in the wrap, $P_{wrap}$. If the wrap is adhesively bonded, then Equations 4–7 can be used to determine $P_{wrap}$ and it can be shown that this is better at confinement at the early stages of the crack development. The same approach can be used to derive the force in the stirrups, $P_{stirrup}$. The confinement, $\sigma_{lat}$, can be derived from $P_{wrap}$ and $P_{stirrup}$. If this is not equal to the initial guess of $\sigma_{lat}$, then it will be necessary to iterate towards a solution. The rest of the procedure for deriving the moment–rotation is unchanged. Hence the unified moment–rotation model can also provide a structural mechanics model for confinement.

An example of a moment–rotation analysis (Farrall et al., 2008) is shown in Figure 44 for a reinforced concrete beam with 1% steel and in which the rotation was limited by wedge sliding. The upper curve is for a 900 mm deep beam and the lower for a 600 mm deep beam; point A is the rotation when the steel yielded, point B for a beam without stirrups or wrap, point C for a beam with stirrups and point D for a beam with both stirrups and FRP wrap. The moment–rotation analysis reflects what is known to occur in tests: deep beams are less ductile than shallow beams; stirrups increase ductility; FRP wrap does not enhance the strength but further enhances the ductility. This provides further evidence of the validity of the generic unified reinforced concrete model.

### 9. Conclusions

It has been shown that empirical models are currently used to quantify the moment–rotation, concrete component of the shear capacity and the effect of lateral confinement. Because of the incredible complexity of reinforced concrete behaviour, these empirical models have been essential in plugging the structural mechanics gaps in our understanding to allow safe design and the introduction of reinforced concrete when it was a new technology. However, these empirical models by their very nature can only be used within the bounds from which they were obtained. They are consequently of limited help in deriving more accurate reinforced concrete design procedures or in allowing the use of new
A generic unified reinforced concrete model has been developed that combines the well-established research areas of SF, PI and RBD to produce a reinforced concrete failure mechanism that can be used for:

(a) any type of reinforcement material such as brittle FRP, ductile or brittle steel, etc.
(b) numerous shapes of reinforcement (e.g. circular ribbed bars, flat externally bonded plates, rectangular NSM plates
(c) both shear and flexural deformations and their combinations.

This generic integrated failure mechanism has been developed into a structural mechanics moment–rotation model that has upper limits due to concrete wedge sliding, reinforcement fracture or debonding. It has also been shown that this generic moment–rotation model can also be used to quantify the effects of shear (and consequently the shear capacity) and the effects of confinement due to stirrups or wraps. It is thus, in effect, a generic unified reinforced concrete model.

The model can explain many phenomena that are known to occur but have eluded structural mechanics solutions. For example:

(a) the reason why the neutral axis depth factor will never truly quantify the moment redistribution because the approach is part of a family of curves
(b) why shear capacity depends on flexural forces and why transverse reinforcement is better than longitudinal reinforcement at enhancing the concrete component of the shear capacity
(c) how compression wedges that form in flexural members as well as in compression members are not an illusion but a SF mechanism that can be quantified, allowing the residual strength of confined concrete to be quantified
(d) how stirrups increase flexural ductility and FRP wrap does not increase flexural strength but does increase ductility.

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